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DISCUSSION OF FORCED VIBRATIONS OF CONTINUOUS BEAMS

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By W. H. Hoppmann, and Edward Saibel and Elio D'Appolonia

ENGINEERING MECHANICS DIVISION

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DISCUSSION

W. H. HOPPMANN[8].—The increasing importance of the knowledge of dynamical resistance of structures makes analytical devices that assist in the determination of deflections and strains highly desirable. The authors are to be commended for their contribution of such a device for use in analyzing the motion of continuous beams. Essentially it consists in treating the intermediate supports (all but the two end ones) by means of undetermined multipliers in setting up the usual Lagrangian equations of motion in generalized co-ordinates. The authors develop the analysis symbolically in some detail. However, they analyze one problem completely. This is a uniform two-span beam subjected to a single concentrated load of sinusoidal time variation. Their formulation of the problem implies a generality of treatment which would really require considerable additional analysis to work out in detail. In this connection the differential equation describing the motion of the beam contains the stiffness and mass as functions of the space variable defining distance along the beam. The numerical example provided as an illustration does not supply the vaguest hint of the additional labor involved if these physical quantities are allowed to vary along the beam.

In the introduction a reference is made to a paper by A. Weinstein,[9] a long study in which approximate frequencies of clamped plates and bars are determined. In the usual fashion these frequencies satisfy certain inequalities. The analysis is made through use of the calculus of variations and on the basis of the observation that there exists a formal mathematical relation of the Lagrangian plate equation to the equation of an elastic membrane, and of the Bernouilli-Euler beam equation to the classical string equation. There are no multiple supports and no undetermined multipliers. A specific remark by the authors as to why they refer to Mr. Weinstein's paper would be interesting. Possibly the writer has missed the point intended.

The applied force assumed by the authors is of the simple sinusoidal type concentrated at a point along the beam. It would be of interest to know what physical loading they had in mind. The writer is aware that it might represent the force produced by a simple vibration generator of the unbalanced inertia type. Also, it might be the force developed by a colliding body as shown by the writer.[10] In this particular case the time that the force was applied is short compared to the fundamental period. The phenomenon is then a transient one; however, the solution is such that it can be applied for the sinusoidal force assumed by the authors. The writer has, also, made an extensive experimental investigation on collisions of solid spheres with one-span, two-span, and three-span beams and compared the strains and deflections with those given by the

NOTE.—This paper by Edward Saibel and Elio D'Appolonia was published in November, 1951, as *Proceedings-Separate No. 100*. The numbering of footnotes in this separate is a continuation of the consecutive numbering used in the original paper.

[8] Associate Prof. of Mech. Eng., The Johns Hopkins Univ., Baltimore, Md.

[9] "Étude des spectres des équations aux dérivées partielles de la théorie des plaques élastiques," by A. Weinstein, *Mémorial des Sciences Mathématiques*, Gauthier-Villars, Paris, France, 1937, pp. 1–58.

[10] "Impact on a Multispan Beam," by W. H. Hoppmann, *Journal of Applied Mechanics*, Vol. 17, No. 4, December, 1950, p. 409.

theoretical solutions.[11] This problem requires consideration of both the motion of the colliding body as well as of the beam. However, it is shown that a sinusoidal force at the point of collision can be used to describe the real force by properly determining the amplitude and duration while satisfying general laws of mechanics. The writer is also aware that the method of generalized co-ordinates and the Lagrangian equations of motion permit ready analysis of cases involving very general forces given as functions of space and time.

In their summary the authors state that further extensions can be made to problems involving damping. It would be interesting to know what law of damping they have in mind. If they mean Stokes' viscous damping then it is not difficult to include it in the general analysis as shown by the writer[12] and others.[13] However, such a law of damping might not be the best to describe the actual physical phenomenon as discussed by R. D. Mindlin,[14] A. M. ASCE. In this event energy expressions for use in determining the Lagrangian equations of motion become rather delicate objects of analysis. In other words, one is faced with a need for determining a proper form of the dissipation function.[15] It is generally conceded that much more attention should be given to damping in connection with beam vibrations. This is a problem demanding theoretical and experimental investigations for possible application in connection with the dynamics of structures.

The authors appear to be concerned solely with deflections rather than strains. This seems to be dictated partly by expediency because of the complexity of the problem. They must solve, simultaneously, a finite number of linear equations each containing an infinite number of terms. They emphasize the rapid convergence of the series. This is not necessarily so for the corresponding strain equations. In fact for transient loading the series are rather slowly convergent. This has been encountered experimentally by the writer.[10,11]

Finally, the authors have not demonstrated a greater economy in numerical calculation using their method rather than solving the differential equation subject to given boundary, continuity, and initial conditions in the classical manner. It would appear that only experience with many actual solutions will answer the question, but the writer suspects on the basis of comparison of what already exists in the way of solutions that the latter method is at least as effective as the one proposed by the authors and possibly more effective.

EDWARD SAIBEL[16] AND ELIO D'APPOLONIA,[17] A.M. ASCE.—As far as the writers could determine at the time of the writing of this paper, the only case

[11] "Impulsive Loading of Beams," by W. H. Hoppmann, *Proceedings of the Society for Experimental Stress Analysis* (publication pending), 1952.

[12] "Impact of a Mass on a Damped Elastically Supported Beam," by W. H. Hoppmann, *Journal of Applied Mechanics, Transactions* A.S.M.E., Vol. 70, 1948, p. 125.

[13] "On the Decay of Waves in Visco-elastic Solid Bodies," by K. Sezawam, *Bulletin of the Earthquake Research Institute*, Toyko Imperial Univ., Vol. III, September, 1927, p. 50.

[14] "Response of Damped Elastic Systems to Transient Disturbances," by R. D. Mindin, F. W. Stubner, and H. L. Cooper, *Proceedings of the Society for Experimental Stress Analysis*, Vol. V, No. 14, 1948, p. 69.

[15] "The Theory of Sound," by Lord Rayleigh, Dover Publishing Co., New York, N. Y., Vol. I, p. 103.

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of a continuous beam under the action of a sinusoidal force was that of uniform beam of two equal spans, a solution of which was published[3] in 1949. If the classical solution were tractable this solution would have been unnecessary. From the fact that there were no such solutions extant, it is obvious that the classical method left much to be desired. In fact, a method that does not permit an actual solution is no solution at all. The solution which the writers present is more general than that cited[3] and can actually be extended to an accuracy as great as desired, in a reasonable length of time. A uniform beam of two unequal spans was selected to demonstrate the technique of applying the method and nothing would have been gained by the reader in studying the solution of a beam with more spans. As for the fact that the writers chose a sinusoidal load, it is evident that using the principle of superposition (applicable because the problem is linear) one can apply the method to any load that can be developed in a Fourier series.

Mr. Hoppmann has indeed missed the point of the method described by Mr. Weinstein.[5] The basic philosophy underlying the treatment by Mr. Weinstein is the development of the solution of a difficult "eigenvalue" problem in terms of the known or easily worked out solution of a different "eigenvalue" problem. However, in the problems treated by Mr. Weinstein the boundary was exterior to the body in all cases. In the case under discussion however, the difficult problem is one in which part of the boundary conditions are given in the interior of the body, and this problem needs to be related to one which has only exterior boundary conditions—a much easier problem. Even in the case of a beam of nonuniform section the writers believe that their method is superior to the classical approach, but they do not wish to minimize the difficulties that may be encountered in this case.

In a solution published elsewhere[18] the stresses created by a moving force on a continuous beam are calculated. Although not as good as that for deflections, the convergence is shown to be quite rapid even without the necessity of using available methods for improving the rapidity of the convergence.[19] The convergence in the case of strains, of course, is of the same order as that of stress.

The writers are aware of the work done by Mr. Hoppmann on the impact of a solid sphere symmetrically applied to a continuous beam of equal span lengths. The fact that the real force between the two can be described by a sinusoidal force makes it indeed a much easier problem by the present method. However, if this were not the case, and if it were necessary to take into account the mass of the impacting body, this can be done by the method presented.[20]

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[3] "Transverse Vibration of a Two-Span Beam Under Action of a Moving Constant Force," by R. S. Ayre, George Ford, and L. S. Jacobsen, *Journal of Applied Mechanics*, A.S.M.E., Vol. 17, No. 1, 1950, p. 1.

[5] "Étude des spectres des equations aux derivées partielles de la théorie des plaques elastiques," by A. Weinstein, *Fascicule 88, Mémorial des Sciences Mathématiques*, Gauthier-Villars, Paris, France, 1937.

[18] "Vibration of a Continuous Beam under a Constant Moving Force," by Winston F. Z. Lee and Edward Saibel, *Journal of the Franklin Institute* (publication pending).

[19] "Partial Differential Equations," by J. D. Tamarkin and W. Feller, "Brown University Notes," Mathematics Dept., Brown Univ., Providence, R. I., 1941, Chapter 3, Section 3.

[20] "Free and Forced Vibrations of Constrained Beams and Plates," by Winston F. Z. Lee, thesis presented to the Carnegie Institute of Technology in Pittsburgh, Pa., in 1952, in partial fulfilment of the requirements for the degree of Doctor of Science.

The type of damping one wishes to consider is immaterial as far as the solution by the present method is concerned. The basic procedure will remain the same. However, what the writers had in mind was the inclusion of damping of a viscous nature—that is, dash-pot damping. Since the theory of the mechanism of internal friction is still in the incomplete stage and undoubtedly is so complicated as to be unmanageable in any differential equation treatment of vibration problems, the writers feel that the method of equivalent viscous damping is sufficient.

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